3 strategies for promoting math disagreements

Engage your students in reasoning and sense making with these effective instructional plans.

By Angela T. Barlow and Michael R. McCrory

As elementary school students attempt to understand mathematical concepts, engaging in the processes of reasoning and sense making is important (Martin and Kasmer 2009–2010). To do so, students should be expected to listen to and challenge their classmates' ideas (Yackel 2001). Disagreements provide students with the impetus to think deeply about mathematics in an effort to make sense of a situation. The discourse that surrounds the disagreement allows students to organize their thoughts, formulate arguments, consider other students' positions, and communicate their positions to their classmates. As differing opinions surface during classroom discussions, teachers receive valuable insights that help them understand children's difficulties with what might otherwise seem like straightforward math. Additionally, as students begin debating mathematical ideas, teachers have occasion to expand students' mathematical thinking.

Recognizing that mathematical disagreements among students engage them in the processes of reasoning and sense making (Yackel 2001), teachers are often willing to be reactive in the sense that if a disagreement occurs, they will allow it to develop and will encourage the discourse that surrounds it. But

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FIGURE 1

After a weekend to think about the role of factors in multiplication, students were given this writing prompt.

4 groups of 8	8 groups of 4
4×8	8×4

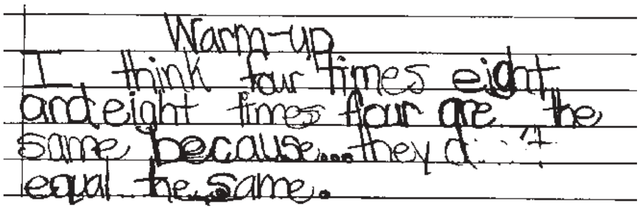
In your journal, complete this statement:

I think 4×8 and 8×4 are/are not the same because . . .

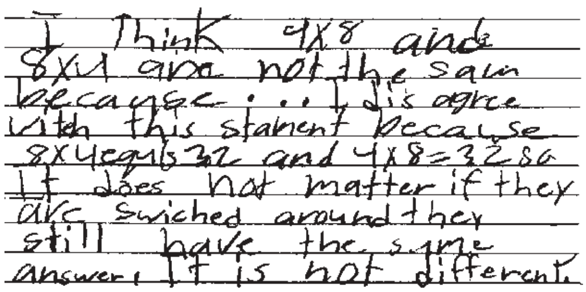
FIGURE 2

Students did writing samples before their multiplication disagreement, expressing their opinions and justifying them.

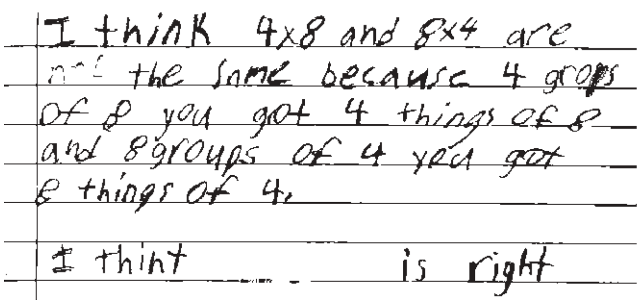
(a) The student self-corrected a journal entry before the classroom discussion.


Warm-up
I think four times eight and eight times four are the same because... they d... equal the same.

(b) This student reflected understanding with a sound argument that supported the mathematical reasoning.


I Think 4X8 and 8X4 are not the sam because . . . I dis agree with this stanent because 8X4 equals 32 and 4X8=32 So It does not matter if they are swiched around they still have the same answer. It is not different.

(c) One student revised his statement after hearing a classmate's argument.


I think 4x8 and 8x4 are not the same because 4 grops of 8 you got 4 things of 8 and 8 groups of 4 you got 8 things of 4.
I think ______ is right

is it possible to be proactive and actually facilitate the development of a disagreement as a means for engaging students in these important mathematical processes?

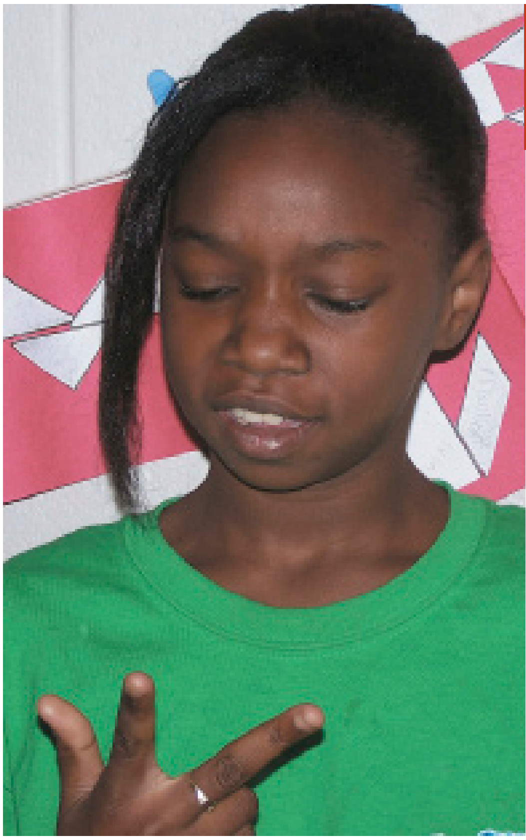
Strategies for encouraging mathematical debate

Our work with a third-grade class revealed three instructional strategies that create opportunities for students to engage in mathematical disagreements in the classroom. The paragraphs that follow describe these strategies, as well as sample tasks or scenarios and student artifacts to support the reader's understanding of each strategy. Additionally, we describe characteristics of issues that lead to disagreements.

1. Force students to choose a side

Writing prompts that force students to pick a side facilitate their engagement in a mathematical disagreement. Choosing sides naturally elicits opportunities for disagreement. We used this strategy to facilitate a discussion during a unit that focused on modeling multiplication. Having begun our multiplication unit by looking at groups of different sizes, students quickly found themselves considering the role of the factors in a multiplication expression. In fact, at the end of class on a Friday, Doug said that 4×8 and 8×4 were the same. Jenaria quickly disagreed. We directed students to think about this disagreement over the weekend. At the beginning of the next class period, they received a writing prompt (see **fig. 1**).

Of three students' initial responses to the prompt, two students indicated that these expressions are the same because they have the same product (see **figs. 2a** and **2b**). The third student wrote that the two expressions are different, basing his decision on the representations of each (see **fig. 2c**). Having the opportunity to write first gave students time to think through their ideas before listening to their classmates. As they displayed their writings via a document presenter, a disagreement arose, engaging students in communicating their reasoning to their classmates. In the initial discussion, the prevalent viewpoint was that the two expressions are the same since they have the same product. In **figure 2c**, the student erased the words *not* and *don't* in his sentence after hearing a classmate's argument. As the class discussion progressed, students began representing the two expressions with equal-sized groups as a means of demonstrating that the two expressions are different. Students in this classroom had often used circles and stars to represent multiplication with equal-



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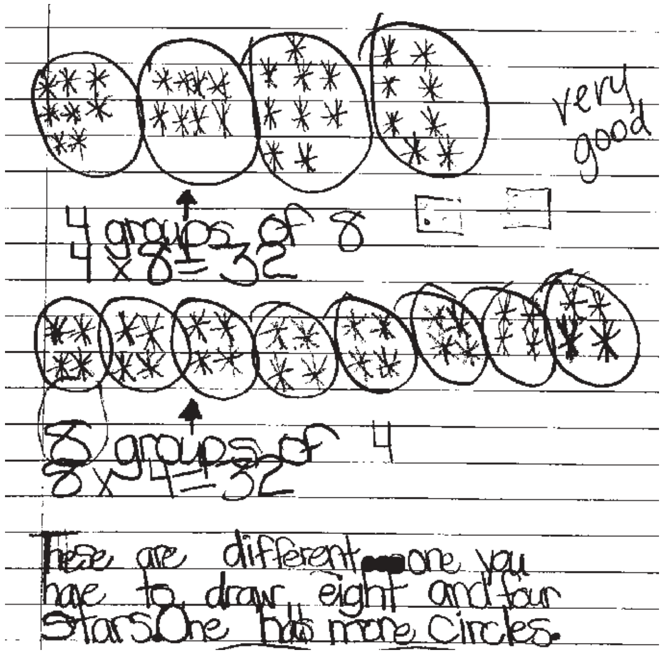
> Giving students opportunities to choose a side and write their ideas sets the scene for math disagreements, dialogue, reasoning, and sense making.

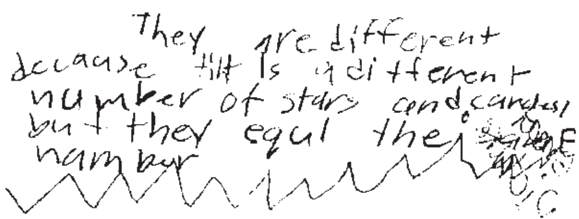
FIGURE 3

Writing samples following the multiplication disagreement show that students now realized that the order of factors does not affect the answer but does affect the representation.

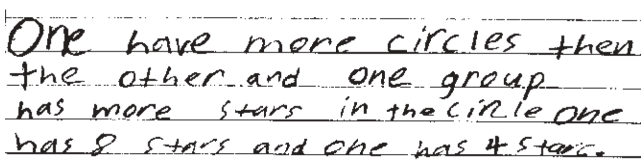
(a)



(b)



(c)



size groups. For example, 4 × 8 would be represented with four circles, each containing eight stars. (See Burns [2001] for an explanation of a game using circles and stars.)

The day after the controversy, students had a chance to respond to the prompt again (see **fig. 3**). In each case, the student recognized the difference between the representations of 4 × 8 and 8 × 4. One student wrote that the two expressions are both different and alike (see **fig. 3b**). After everyone had completed their writing, randomly selected students once again displayed their writings via the document presenter. As a result of this disagreement, students realized that the order of the factors does not impact the answer; but it does impact the representation.

In the previous multiplication example, the relationship of the two expressions provided two seemingly opposing sides to the issue (i.e., either the two expressions are the same, or they are different), furnishing students with the chance to state and support their reasoning. The writing prompt gave all students the chance to think about the issue and select a stance to take before the disagreement ensued. Two similar writing prompts follow:

- **Mrs. Jones** gave her students the equation 7 + 8 = ___ + 5 and asked them to tell what number should go in the blank to make the equation true. Mary said that a 15 should go in the blank. Keyon said that a 10 should go in the blank. Who do you agree with, and why?
- **Jamie** said that a square is a rectangle. Do you agree with Jamie? Why, or why not?

Once students have responded to the writing prompt, they should have opportunities to share and support their reasoning, thus engaging in the disagreement.

2. Reveal students' misconceptions

Tasks that are designed to address students' misconceptions provide the opportunity for disagreements to arise. As an example, consider the broken-ruler task, which is intended to reveal students' misconceptions regarding measurement of length. In this task, students were given a paper ruler (see **fig. 4**) and asked to use it to measure the long side of an index card. Having read about using a broken ruler (Barrett et al. 2003), we anticipated that the task would lead to a disagreement surrounding the index card's length, as some students would report its actual length (five inches), but others would misread the ruler. As the lesson unfolded, three separate disagreements arose. First, some students reported that the index card measured five-and-a-half inches *and* six inches, leading to a disagreement over whether the long side of the index card could have two different lengths (see **fig. 5**). Second, as students provided their arguments regarding the possibility of two different lengths, a disagreement arose over whether to count the lines on the ruler or the spaces between the lines. Finally, a disagreement surfaced over where to line up the index card, either at the initial line or at the end of the ruler.

Examining students' responses to this example reveals that using a task designed to uncover their misconceptions prompts disagreements that give students the chance to share their ideas and defend their reasoning. Student discussions generated by the disagreement reveal an interesting possibility: When measuring lengths, students can actually have different results if dissimilar units are used in the measuring process. According to Karl, "That's the only way you can get two different measurements—using centimeters and inches." Such a revelation facilitated thinking about the underlying assumption of the disagreement, that we were all using the same units.

In a task adapted from Schifter (1999) and Clements and Sarama (2000), students identify which figures are triangles and give a description of how they know whether the figure is a triangle (see **fig. 6**). On the surface, any disagreement that comes from this task may appear to be centered on math vocabulary or the definition of a triangle.

FIGURE 4

Such tasks as using a broken ruler for measuring are designed to uncover misconceptions and elicit disagreements.

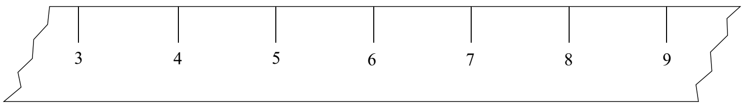


FIGURE 5

Had disagreements between students not occurred, gaps within their understanding most likely would not have been discovered. As students demonstrate their viewpoints to their peers, they make connections between their ideas, and they resolve other false ideas. (Teacher prompts are in bold text.)

[*Frank*] There's two answers.

Is it possible for something to have two different lengths? Outside of a piece of gum that can be stretched . . . If you can't stretch it, is it possible for something to have two different lengths?
[*Jenaria*] Like what Brianna said; like you can either round, and it can go up to six inches, or you don't have to round. You can just leave it like it is at five-and-a-half inches, or you can round it and have it be six inches.

[*Karl*] Yes, if you do it like that. [*He motions to two sides of the index card, pauses, then clarifies that the students were just measuring the long side of the card.*] Oh, no. Because the ruler only can be like [*pauses*] if you do it like that, you've still got the same thing [*pointing toward the numbers on the ruler*]. Only if you turn it around.

So that changes the length of the card when you move the ruler.
[*Karl*] Not exactly.

[*Using the broken ruler to demonstrate at the front of the class, Frank aligns the end of the index card with the end of the ruler.*] Now, there's 1, 2, 3, 4, 5, and a half. [*Afterward, he slides the index card so that it aligns with the three-inch mark.*] But now, um, there is 1, 2, 3, 4, 5, 6. [*Note that Frank is counting the inch marks on the ruler and not the spaces between them.*] You didn't need to move the ruler, because this is in centimeters.

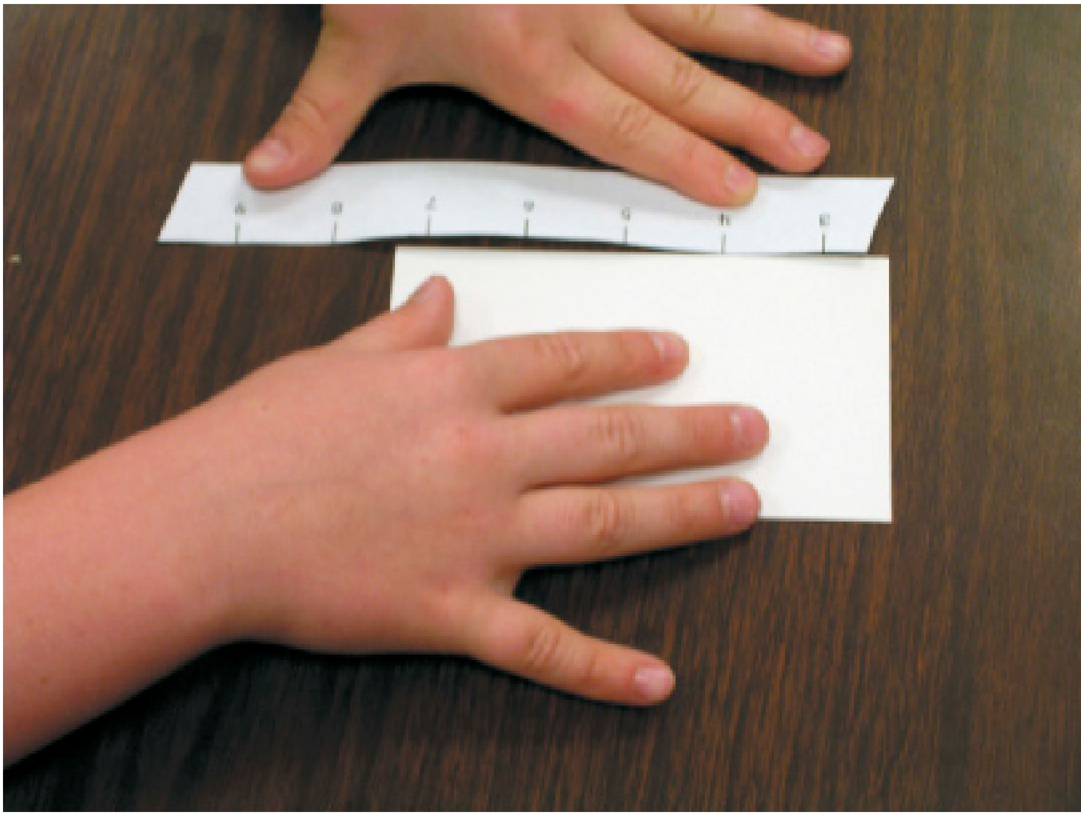
[*Karl*] [I disagree with] him doing the centimeters. We are not learning about centimeters. That's mostly the only way you can do it two times. That's the only way you can get two different measurements—using centimeters and inches.

[*Ayona*] If he changes the ruler to here [*aligning the edge of the index card with the edge of the broken ruler*], then you have to count this part too [*motioning to the initial part of the index card before the three-inch mark*].

[*Frank*] Yes, I was counting this [*pointing toward the part of the card after the final inch mark*]. It's a half.

[*Ayona*] But you weren't counting this [*pointing toward the initial part of the index card before the three-inch mark*].

[*Frank*] Oh, yeah.



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Discussing the broken-ruler task revealed the unspoken assumption that everyone was using the same units of measurement (inches).

Elementary school students, however, have not typically reached the descriptive level, where they can characterize shapes by their properties. So they rely on precognition, or visual-level thinking, where shapes that "look" like triangles can be classified as triangles (Clements and Sarama 2000). Thus, without the students having made this transition from precognition, or visual-level thinking, to a descriptive level of thinking, this task helps students develop their own understandings of what a triangle is and is not by allowing them to disagree on particular figures, formulate mathematical arguments for their points of view, and carry out a debate until the figure can be properly classified and agreed on.

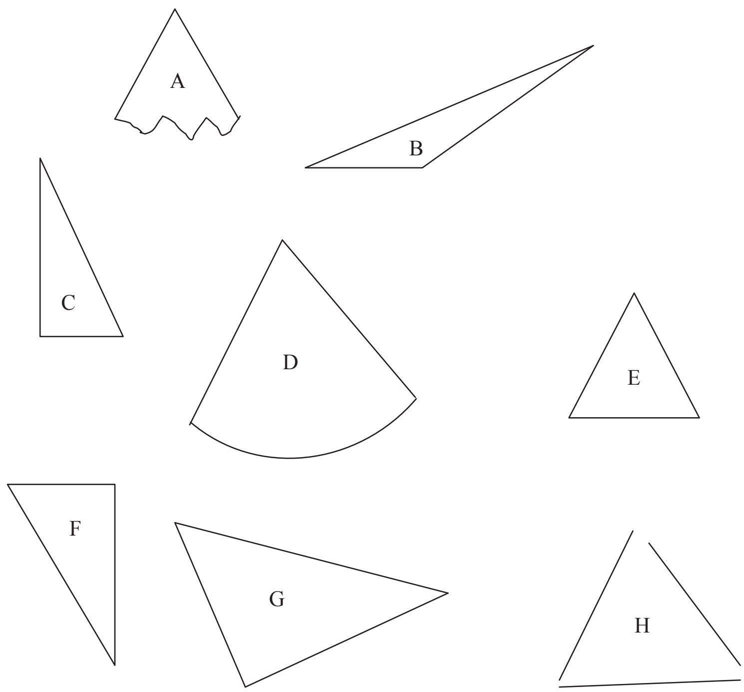
3. Recall last year's disagreements

A third strategy for facilitating mathematical disagreements is to purposefully design and ask questions that have led to disagreements in your previous classes. For example, because the third graders in our classroom had experience representing and decomposing three-digit numbers with base-ten blocks, we posed what seemed like a natural question for students to consider (see **fig. 7**). The disagreement occurred when students had to decide whether the ten flats were equivalent to ten hundred or one thousand, an issue that arose through sharing their ideas. For adults, ten hundred versus one thousand may not seem like an issue at all. In fact, one might

FIGURE 6

The triangle-classification task adapted from Schifter (1999) and Clements and Sarama (2000) prompts students to debate until they can properly classify a figure and agree on it.

Directions: Examine each two-dimensional figure below. Draw a circle around the figures that are triangles. At the bottom of the page, write a description of how you can tell whether a figure is a triangle.



feel it is just a question of syntax or vocabulary. With this view, the adult's tendency would be to say that ten flats represent one thousand, which has a value of ten hundred. Doing so would not help students like Kendra, though, who see one thousand as being much larger than ten hundred. The issue is not in how to read the number 1000. Instead, the mathematics underlying this issue includes understanding that a value can be represented and expressed in multiple ways as well as understanding that the structure of the base-ten system allows for ten of one unit (in this case, ten hundreds) to be equivalent to one of another unit (in this case, one thousand). These ideas around place value are key mathematical concepts and therefore worthy of attention.

In the example in **figure 7**, James's reasoning convinced the class that one thousand and ten hundred are equivalent. Follow-up tasks and discussions over the next few days engaged students in representing four-digit numbers in multiple ways, further building on these ideas.

Prior to asking the question, we really did not anticipate a disagreement. Looking back, however, we now understand why one arose. We are thankful that the question facilitated its occurrence. Now that we are aware of this issue, we can plan to use this same question in future classes, recognizing the importance of the disagreement that it will most likely invoke.

FIGURE 7

These third graders had previously represented and decomposed three-digit numbers with base-ten blocks. Arguments arose around place value—a key mathematical concept that is worthy of attention. (Teacher prompt is bold.)

If ten units can be put together to form one long, and ten longs can be put together to form one flat, what do we have when we put ten flats together? [*The teacher instructed students to think inside their heads and then share with a partner*.]

[*Alan*] You think it's ten hundred; but it can't go above ten hundred, so I'd have to all go to one thousand. So, I mean, it would start over but at a higher level, one thousand.

[*Doug, standing at the board*] The way how I did it, it had these zeroes and one 1 [*writing* 1000 *on the board*]. And, and, and after [*trying to get the other students' attention*], look. And after there are three numbers, I put a comma [*placing a comma between the* 1 *and the first zero*]. That's what I thought. Because, if it—if he thought it was ten hundred [*erasing the comma*], that wouldn't make sense. Nope, that would be wrong [*Xing out the* 1000 *on the board and writing* 1,000 *out to the right*]. But, this is right [*putting a check mark by the* 1,000].

[*Kendra*] I want to say I disagree with all of them, because if you count on your fingers, you're gonna get one hundred, two hundred, three hundred, four hundred, five hundred, six hundred, seven hundred, eight hundred, nine hundred, ten hundred. How in the world can you skip from all the way to ten hundred? From one hundred to a thousand? When you count on your fingers, when you count on your fingers, you're going to get to ten hundred, and that's how I know, because I know I can't skip all the way from one hundred to a thousand. I ain't got that many fingers.

[*James*] Um, I thought that it's, it's actually both of them because, because ten equals [*pausing to look at* 1000 *and* 1,000 *on the board*] all of the things that had changed for both of them was that comma because they've got the, they've got the same number of zeroes and one. The only thing that would really actually change was if he putted that comma right there after the one. Then it would have been the same.

Issues that lead to disagreements

As teachers who have purposefully planned to engage students in mathematical debate, we have identified three significant characteristics of issues that facilitate students' engagement in disagreements in elementary school classrooms.

1. Center on a mathematical concept

The issue must center on a mathematical concept. Issues such as syntax, math vocabulary, and procedures tend to be supported by the basic rules of mathematics and should not be up for debate. On the other hand, a group of students often do not understand math concepts in the same way; these differences can lead to a situation where a disagreement arises. This type of situation gives the teacher a prime opportunity to allow students to clear up their own misunderstandings by internalizing the results of the discussion and assimilating this new, self-remediated understanding into their mathematical cognition. For example, the disagreement during the triangle-identification task causes students to evaluate their understanding of what constitutes a triangle and develop those characteristics into a working definition of a triangle. These changes in thinking will result in the migration of students' thinking processes from precognition and visual levels to the descriptive level.

2. Are accessible to all

Second, the issue must be accessible to all students in the classroom. Regardless of their mathematical misunderstandings around the disagreement, students should have some true understanding that will serve as a foundation for their position on the disagreement. This foundation allows students to engage in the math being debated and provides the teacher with a common point of knowledge among the students as he or she facilitates the discussion. In the multiplication example, students' access to a means for representing multiplication allowed them a point of entry for investigating the role of the factors in a multiplication expression. Although concrete materials were not used in this case, often the availability of manipulatives can provide this accessibility for some students who might otherwise not have access.

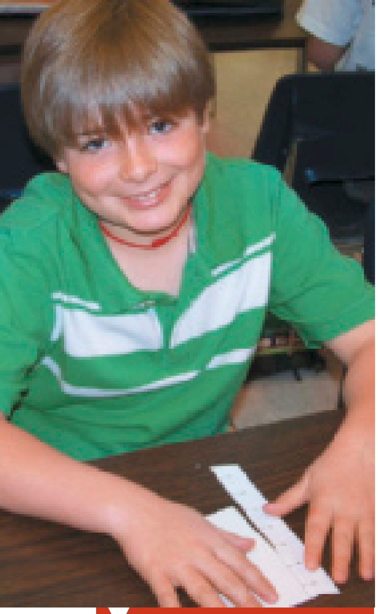
3. Can be debated

Third, students must be able to debate the issue. If, for example, the disagreement comes in the form of two solutions that contradict (or seem to contradict) each other, students should view the situation as problematic. This should lead them to choose a side of the disagreement and formulate a mathematical argument to prove their viewpoint. The situation could also offer students a chance to weigh both sides if they are unsure of which side is correct.

Students took either side of the argument in the ten hundreds versus one thousand disagreement. Some students were unsure of which side was correct. Although a few students straddled the issue, unable to decide on a position to take, every student was able to actively participate in the debate through argumentation or self-reflection.

The teacher's role

Teachers play a key role in the success of engaging students in mathematical disagreements. Before the disagreements, the teacher must work to establish a classroom environment that values risk taking, open discussion, and debating ideas. Students must learn to value and respect one another's opinions to the point that *all* students feel comfortable contributing to the discussions. If this environment has not been set up, disagreements can be negative, frustrating



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The proper classroom environment is key to building students' thinking and their participation, as well as avoiding negative disagreements.

experiences for students. If, on the other hand, students have become accustomed to analyzing student work and agreeing or disagreeing with discussion points, then disagreements provide opportunities to build students' thinking.

During the disagreement, the teacher has a two-fold responsibility in facilitating the discussion. First, when they initially engage in disagreements, students will have the inclination to share their thoughts and opinions without really considering the reasoning expressed by their classmates. The sample dialogues we have provided do not demonstrate this tendency, as they were taken later in the school year. We did have to address these tendencies, however, so that students would begin to actively listen to one another. To facilitate this change in behavior, we used questions such as these: Can you repeat what Sarah just said? Do you agree with what Marcus said? How is what Marcus said different from what Sarah said?

The second aspect of teacher responsibility is to fight the urge to participate during the discussion. Although refraining from sharing what you think—or from telling students what the correct answer is—can be extremely difficult, teachers risk losing valuable opportunities for learning to occur. Instead of thinking deeply about what has been said, sharing their reasoning, making sense of what others say, or revealing their understandings and misunderstandings, students will look to the teacher to resolve the disagreement and assume that the teacher is correct.

After the disagreement, the teacher must decide how to proceed:

- **What if** students are able to come to a mathematically correct consensus? Then the teacher must follow up with a question or task that uses that knowledge, allowing the issue to arise again, if necessary.
- **What if** students cannot come to a consensus? Then the teacher must design a follow-up task that will provide them with a deeper understanding of the mathematics necessary for resolving the disagreement.
- **What if** they all agree on an idea—and the idea is wrong? Then the teacher must design a follow-up task that will force students to confront their misunderstanding.

Regardless of whether students reach a correct consensus or disagree with one another, following up on this work is key to developing the type of mathematical understandings that we want our students to obtain.

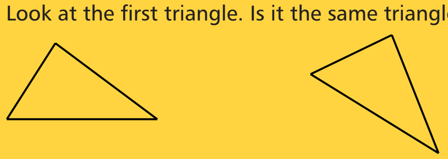
Specific questions that lead to disagreement

Consider questions (followed by justification) that will most likely lead to a mathematical debate among your students:

1. Is this equation true? 25 = 25

Students who believe this is a false statement will assert that you cannot have just a 25 on the left-hand side of the equation. For more information on this operational view of the equal sign, see Carpenter, Franke, and Levi (2003).

2. Look at the first triangle. Is it the same triangle if I turn it?



Original triangle   Turned triangle

Students who base their decision on how the triangle looks will say the original triangle has one horizontal side and two diagonal sides, whereas the turned triangle has three diagonal sides. For more information, see Clements and Sarama (2000).

With each of these questions, it may be the case that no disagreement occurs as all students supply the correct answer and support it with mathematically accurate justifications. If, however, such a question caused a disagreement with your students in past classes, it will most likely lead to a similar debate with your current students.

Conclusion

Is 4 × 8 the same as 8 × 4, or are they different? Is it possible for the long side of an index card to measure both five-and-a-half inches *and* six inches? Do ten flats represent ten hundred or one thousand? As teachers, we most likely use writing prompts, tasks, and questions similar to those presented here. But we do not always recognize the potential these strategies have for evoking mathematical disagreements. As the previous examples demonstrate, creating opportunities for disagreement offers students the chance to engage in reasoning and sense making. By establishing the appropriate classroom environment and ensuring that all engage in the discussions, teachers encourage students to develop mathematical understanding and to thereby benefit from mathematical disagreements.

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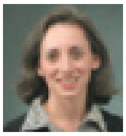

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