

Describing Levels and Components of a Math-Talk Learning Community

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The transformation to reform mathematics teaching is a daunting task. It is often unclear to teachers what such a classroom would really look like, let alone how to get there. This article addresses this question: How does a teacher, along with her students, go about establishing the sort of classroom community that can enact reform mathematics practices? An intensive year-long case study of one teacher was undertaken in an urban elementary classroom with Latino children. Data analysis generated developmental trajectories for teacher and student learning that describe the building of a *math-talk* learning community—a community in which individuals assist one another’s learning of mathematics by engaging in meaningful mathematical discourse. The developmental trajectories in the Math-Talk Learning Community framework are (a) questioning, (b) explaining mathematical thinking, (c) sources of mathematical ideas, and (d) responsibility for learning.

Key words: Classroom interaction; Pedagogical knowledge; Reform in mathematics education; Teaching (role, style, methods), Teaching practice

The successful implementation of mathematics education reform requires that teachers change traditional teaching practices significantly, and develop a discourse community in their classroom (National Council of Teachers of Mathematics [NCTM], 2000). Yet the prospect of creating such a community is daunting to many teachers; they often do not know where to begin to create the kind of discourse practices described by NCTM. The goal of this article is to introduce a framework that can help to guide teachers’ work in this area and to facilitate researcher and teacher educator understanding of this process.

Over the past decade, numerous studies have investigated teachers’ attempts to change their mathematics instruction in light of the goals of reform (e.g.,

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Cohen, 1990; Fennema & Nelson, 1997; Wood, Cobb, & Yackel, 1991). Some of this work highlights the need for increased subject matter knowledge and pedagogical content knowledge on the part of teachers and, in particular, the importance of providing opportunities for teachers to learn about student thinking (Fennema et al., 1996). Other research describes the many dilemmas that teachers face in trying to implement reform, and more specifically in establishing a discourse community. For example, teachers may find that students disengage somewhat as they use more challenging tasks (Romagnano, 1994; Stein, Grover, & Henningsen, 1996). In other cases, as teachers open up their classroom for students' ideas, they find it more difficult to manage the mathematical direction that instruction takes or find that students are making claims that are mathematically incorrect (Jaworski, 1994; Sherin, 2002a; Silver & Smith, 1996). A third dilemma involves teachers' sense of efficacy (J. P. Smith, 1996). Teachers find that in the context of reform, it is much more difficult to predict where a lesson will go and thus more difficult to anticipate and prepare for their role in instruction (Heaton, 2000; Sherin, 2002b; M. S. Smith, 2000). Thus, although the development of a discourse community is seen as a critical step in the implementation of reform, teachers may face difficulties and dilemmas as they make this transition.

In this article, we address this issue by introducing a theoretical framework that elaborates the development of a *math-talk learning community*. By math-talk learning community, we refer to a classroom community in which the teacher and students use discourse to support the mathematical learning of all participants. A primary goal of such a community is to understand and extend one's own thinking as well as the thinking of others in the classroom. The framework we offer extends prior research on teacher change in the context of reform by describing key components of a math-talk learning community as well as the intermediary levels along which the community develops. This description seeks to provide teachers with steps to develop their classroom into a rich math-talk learning community. Such step-by-step changes can affect classrooms on a large scale, particularly if passage through the steps can also be supported by reform-based curriculum materials (Ball & Cohen, 1996).

This article is based primarily on a case study of one teacher who began the year by teaching in a traditional manner in her urban Latino third-grade classroom. Over the course of the year, however, she had considerable success in implementing mathematics education reform, particularly in the area of whole-class discourse. Many educational reforms bypass classrooms with children from poor or non-English speaking backgrounds (Spillane, 2001) partly because such children are assumed not to be linguistically prepared to participate in reform-based practices. Thus, success in this case is particularly significant for it supports the notion that urban classrooms with students that are below grade level in mathematics *can* function and learn as a math-talk learning community.

PERSPECTIVES ON TEACHING AND LEARNING

Two positions anchor the perspectives on teaching and learning that frame the study reported here. First, a Vygotskian viewpoint, as articulated by Gallimore and Tharp (1990) suggests that teaching is beneficial when it “awakens and rouses to life those functions which are in a stage of maturing, which lie in the zone of proximal development” (p. 177). Thus, learning occurs when assistance is provided at opportune points in the learner’s zone of proximal development. Furthermore, Vygotsky’s notion of movement from the inter-psychological to the intra-psychological plane characterizes performance as moving from being assisted to being independent over time. In this study, both the teacher and the students moved through their own learning zones of proximal development. Moreover, they assisted one another in a recursive process as they moved through various levels of development. In this article, we describe the kinds of assistance provided to the students as they successfully internalized new roles in the math-talk learning community. Although the focus teacher was also provided with various means of assistance in her development (e.g., researcher interviews, implementation of a research-based curriculum, teacher meetings, and supportive administrators), describing these means of assistance is beyond the scope of this article (see Hufferd-Ackles, 1999). Instead, what is central here is the description of changes in teacher and student interactions as they moved together through learning zones of each new level of the math-talk learning community.

The other perspective that anchors this research is a constructivist and socioconstructivist view of learning (e.g., Cobb, 1994; Cobb & Bauersfeld, 1995; Cobb, Wood, & Yackel, 1990; Cobb, Yackel, & Wood, 1993). This socioconstructivist epistemology blends radical constructivism (von Glasersfeld, 1990) and sociocultural perspectives. From a radical constructivist’s perspective, learning is about self-organization. Social construction of knowledge is related to a Vygotskian perspective and asserts that an individual’s learning is affected by participating in a wider culture, the classroom, and the outside world (e.g., Cobb, 1994). For example, *taken-as-shared* mathematical meanings are constructed through a process of interacting in a community; these meanings become cultural representations and norms for interacting (Cobb & Bauersfeld, 1995). What is critical for our research is the notion that in a constructivist classroom, participants consider all members of the community to be constructing their own knowledge and reflecting on and discussing this knowledge.

METHOD

Participants

Four teachers from St. Peter Elementary School¹ participated in this study during the 1997–1998 school year. St. Peter is a Catholic school located in a working-class, Latino section of a large U.S. city. Ninety-eight percent of students receive schol-

¹ Pseudonyms are used throughout this article for the school, teacher, and students.

arships toward their tuition from the parish and broader Jesuit-sponsored fundraisers. Three of the teachers were in their second year of teaching, and one teacher had no prior teaching experience. Two of the teachers were female, and two were male; two of the teachers were Latino, and two were European-American. The majority of the children in the school spoke English as their second language and Spanish as their first language. The school had one class at first, second, third, and fourth grade. The four teachers each taught one of these grades in self-contained classrooms.

As will be explained shortly in the article, the third-grade classroom became the focus of a case study. This teacher had taught for 1 year previously, and she and her students moved from second grade to third grade together. Her class of 25 students represented a wide range of achievement levels based on their performance during the previous year. Spanish was the language spoken by all of these students at home, though many students' grasp of English was also fairly strong. The teacher was bilingual and consistently monitored student comprehension of language. The students sometimes slipped into Spanish when they were excited about something or when they were working particularly hard to be understood.

Curriculum

As part of the study, the four teachers implemented the research-based mathematics curriculum, Children's Math Worlds (CMW) (Fuson et al., 1997). The CMW curriculum is based on years of research into the manner in which children learn and understand number concepts. CMW contains key conceptual supports including language and representations that help mathematics to become personally meaningful to students and that provide a context through which students can share their ideas with others. This curriculum suggests that students make mathematical drawings to solve problems and explain their thinking and label these drawings and related equations to link them to the problem situation. Because the students at St. Peter were not learning mathematics in their first language, such visual referents were particularly important. In addition, the curriculum provides support for students to use alternative methods of solving problems. It also supports teacher understanding of these alternative methods by providing information about predicted students' response to a range of activities. CMW emphasizes the building of a learning community and of meaning making for both student and teachers.

The CMW curriculum was the designated mathematics curriculum at St. Peter School for first through third grades and for part of the fourth grade. Because the teachers in this study were using instructional tasks from the curriculum, they were able to concentrate on the development of their practice rather than on the development of instructional tasks that may or may not have offered students significant opportunities to extend their mathematical thinking (e.g., Wood, Cobb, & Yackel, 1991). Prior to the teacher's departure for maternity leave, her third-grade class completed the units on two major topics: single-digit multiplication and division and multidigit addition and subtraction.

Classroom Observations

The four teachers were observed throughout the year, although each was on a slightly different observation schedule. The first-grade class was observed either once per week or every other week; the second-grade class was observed twice per week from November to February and once per week thereafter through June; the third-grade class was observed twice per week from September to mid-April (at which point the teacher left for maternity leave); and the fourth-grade class was observed twice per week in the fall and every other week in the spring. Most observations were videotaped, and those that were not were audiotaped. Following each observed class, a postobservation interview was conducted individually with each teacher with the exception of the first-grade teacher. Because the first-grade classes were conducted in Spanish, these lessons were videotaped for a Spanish-speaking researcher to analyze. That researcher conducted telephone interviews with the teacher.

Two researchers conducted most classroom visits to the second-, third-, and fourth-grade classes. One researcher videotaped the mathematics lesson, and the other took detailed notes. The priorities of the videographer in the classroom were to follow the teacher or other speaker and to record all student work on the board. For the observations that were not videotaped, there was one researcher in the room taking notes, and the lessons were audiotaped. The tapes provided permanent records for later analysis.

The first priority for note taking was to follow the teacher and document as many of his or her actions and words as possible. Notes were made of what happened during each segment of the class, important teacher and student conversations, questions, and statements, all student board work, noteworthy instructional practices, and classroom social climate (e.g., how many students raised their hands to respond to the teacher or another student). The note-taking researcher had several years of classroom teaching experience that were helpful in understanding the complexities of the teacher's role and attending to multiple simultaneous events (see Day, 1988).

During the following year, in the classroom of the third-grade teacher, seven classroom observations were conducted over the first 2 months of school. These visits focused on the formation of a math-talk learning community with the new class of students. Postobservation questions focused on the teacher's understanding of the development of this community in her classroom and on her continued math-content and math-pedagogy learning.

Teacher Meetings

All teachers met together twice monthly to discuss their mathematics teaching. These meetings were initiated by the St. Peter administrators, the principal and assistant principal, who were advocates and catalysts for reform in all subject areas. The field researcher facilitated these meetings throughout the year. Each meeting was videotaped or audiotaped.

DATA ANALYSIS

Phases

Data analysis consisted of three main phases. The first phase of analysis occurred during the data collection period and informed the data collection process (Miles & Huberman, 1984). Throughout this time, the field researcher, who is the first author of this article, met regularly with the other researchers to discuss the detailed observations notes that were available from the classrooms. The goal at this point was to identify significant changes that were occurring across and within the classrooms. Three researchers read the field notes independently; thus, the meetings served as a form of investigator triangulation (Denzin, 1984). Investigator triangulation also took place as data were examined in light of current literature on teacher learning and mathematics reform as well as of ongoing research on the use of CMW (see Fuson, Smith, & Lo Cicero, 1997; Fuson et al., 2000) and of the reform-based curriculum *Everyday Mathematics*². Working hypotheses were examined as data analysis and data collection interacted (e.g., Spillane, 2000), and ensuing observations and interviews were modified to pursue issues as they were identified.

Based on this analysis, it was determined that the practice of the third-grade teacher, Ms. Martinez, had exhibited dramatic change over the course of the school year. Although there were positive changes in the direction of reform in each of the teachers' practices, Ms. Martinez's class showed the most change. It began as very traditional and moved to a fully implemented mathematical discourse community. This classroom was therefore selected as the focus of a case study.

The second phase of analysis consisted of a case study of the third-grade teacher, Ms. Martinez. This involved an analysis of classroom discourse, teacher interviews, and teacher meetings based on verbatim transcriptions of videotaped and audiotaped recordings. Transcriptions from recordings described as accurately as possible all spoken words from classroom observations. In addition to dialogue, the videotaped transcriptions contained descriptions of behavioral contexts.

There are trade-offs inherent in the use of the case-study method: in-depth understanding is gained while generalizability may be lost. To address the issue of generalizability, we added a third phase of analysis. This involved examining the results of the case study within the context of data collected in the other three classrooms. Additional observations were also conducted during the following school year to further examine the robustness of the findings. The description of the framework was modified to reflect observations in the other classrooms and in the second year to resonate with observations of other CMW classrooms and of classrooms in the *Everyday Mathematics* longitudinal study.

² The second author of this article was simultaneously conducting an empirical study of the implementation of *Everyday Mathematics* (see Mills, 1996; Mills, Fuson, & Wolfe, 1999).

The data summarized in this article enable a detailed look at longitudinal growth across a school year. Moreover, this rich data set can help to provide the reader with an in-depth look at and understanding of the synergistic classroom life that led to the framing of a developmental trajectory that can subsequently be applied to other classroom situations (Brown, 1992; Donmoyer, 1990).

Establishing the Framework

In order to begin to classify and organize the large amounts of data collected in the case-study classroom, we established a coding system. Initially, classroom observation notes, transcripts from the classes, and teacher postobservation interviews were classified in light of a variety of themes related to mathematics reform. These were organized chronologically, with the lesson considered to be the unit of analysis. Within the lessons, examples of dialogue from classroom transcripts that had a clear beginning and end were designated as *episodes*. Each of the 60 classroom transcripts contained approximately 8 to 10 episodes. Three themes and the relationships among them soon emerged as central, and these became the focus of data analysis: *evidence of mathematics community*, *teacher actions*, and *student actions*. Establishing the themes as the focus illustrated that the development of the mathematics community was *linked* to specific teacher actions and/or student actions. That is, as students responded to particular kinds of actions by the teacher, the class more and more reflected ideals of mathematics reformers.

Within these actions, we identified four distinct, but related components that captured the growth of the math-talk learning community over time, and we followed their growth in the data: (a) Questioning, (b) Explaining math thinking, (c) Source of mathematical ideas, and (d) Responsibility for learning. Within each attribute, developmental trajectories in teacher actions and students' actions were derived from the data. By developmental trajectory, we refer to changes in the teacher's and students' actions that occurred over time and built successively on one another. Each trajectory consists of four levels—Level 0 through Level 3. Together, these four trajectories represent the development of the math-talk learning community in Ms. Martinez's classroom. The resulting framework titled *Levels of the Math-Talk Learning Community: Action Trajectories for Teacher and Student* is shown in Table 1.

The articulation of the Levels of Math-Talk Learning Community framework went through cyclical revisions. The revision process continued until all episodes from all lessons in the data fit within a cell of the framework. In addition, triangulation with data from the other three St. Peter teachers (i.e., placing episodes from their transcripts in the framework) as well as with the data from the Everyday Mathematics classroom study (Mills, 1996; Mills, Fuson, & Wolfe, 1999) enabled further modifications of the Levels of Math-Talk Learning Community framework and provided confirming analysis. To check interrater reliability of the categories, another coder coded 13 classroom sessions chosen randomly from the whole course of the study (about one class every 2 weeks). Interrater agreement was 100%

Table 1
Levels of the Math-Talk Learning Community: Action Trajectories for Teacher and Student

Overview of Shift over Levels 0–3: The classroom community grows to support students acting in central or leading roles and shifts from a focus on answers to a focus on mathematical thinking.	
A. Questioning	D. Responsibility for learning
<p>Shift from teacher as questioner to students and teacher as questioners.</p> <p>Students increasingly explain and articulate their math ideas.</p>	<p>Shift from teacher as the source of all math ideas to students' ideas also influencing direction of lesson. Math sense becomes the criterion for evaluation.</p> <p>Students increasingly take responsibility for learning and evaluation of others and self.</p>
Level 0: Traditional teacher-directed classroom with brief answer responses from students.	
A. Questioning	D. Responsibility for learning
<p>Teacher is the only questioner. Short frequent questions function to keep students listening and paying attention to the teacher.</p> <p>Students give short answers and respond to the teacher only. No student-to-student math talk.</p>	<p>Teacher repeats student responses (originally directed to her) for the class. Teacher responds to students' answers by verifying the correct answer or showing the correct method.</p> <p>Students are passive listeners; they attempt to imitate the teacher and do not take responsibility for the learning of their peers or themselves.</p>
B. Explaining mathematical thinking	C. Source of mathematical ideas
<p>No or minimal teacher elicitation of student thinking, strategies, or explanations; teacher expects answer-focused responses. Teacher may tell answers.</p> <p>No student thinking or strategy-focused explanation of work. Only answers are given.</p>	<p>Teacher is physically at the board, usually chalk in hand, telling and showing students how to do math.</p> <p>Students respond to math presented by the teacher. They do not offer their own math ideas.</p>

Level 1: Teacher beginning to pursue student mathematical thinking. Teacher plays central role in the math-talk community.

<p>A. Questioning</p> <p><i>Teacher questions begin to focus on student thinking and focus less on answers. Teacher begins to ask follow-up questions about student methods and answers. Teacher is still the only questioner.</i></p> <p>As a student answers a question, other students listen passively or wait for their turn.</p>	<p>B. Explaining mathematical thinking</p> <p><i>Teacher probes student thinking somewhat. One or two strategies may be elicited. Teacher may fill in explanations herself.</i></p> <p>Students give information about their math thinking usually as it is probed by the teacher (minimal volunteering of thoughts). They provide <i>brief descriptions</i> of their thinking.</p>	<p>C. Source of mathematical ideas</p> <p><i>Teacher is still the main source of ideas, though she elicits some student ideas. Teacher does some probing to access student ideas.</i></p> <p>Some student ideas are raised in discussions, but are not explored.</p>	<p>D. Responsibility for learning</p> <p><i>Teacher begins to set up structures to facilitate students listening to and helping other students. The teacher alone gives feedback.</i></p> <p>Students become more engaged by repeating what other students say or by helping another student at the teacher's request. This helping mostly involves students showing how they solved a problem.</p>
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Level 2: Teacher modeling and helping students build new roles. Some co-teaching and co-learning begins as student-to-student talk increases. Teacher physically begins to move to side or back of the room.

<p>A. Questioning</p> <p><i>Teacher continues to ask probing questions and also asks more open questions. She also facilitates student-to-student talk, e.g., by asking students to be prepared to ask questions about other students' work.</i></p> <p>Students ask questions of one another's work on the board, often at the prompting of the teacher. Students listen to one another so they do not repeat questions.</p>	<p>B. Explaining mathematical thinking</p> <p><i>Teacher probes more deeply to learn about student thinking and supports detailed descriptions from students. Teacher open to and elicits multiple strategies.</i></p> <p>Students usually give information as it is probed by the teacher with some volunteering of thoughts. They begin to stake a position and articulate more information in response to probes. They explain steps in their thinking by providing <i>fuller descriptions</i> and <i>begin to defend</i> their answers and methods. Other students listen supportively.</p>	<p>C. Source of mathematical ideas</p> <p><i>Teacher follows up on explanations and builds on them by asking students to compare and contrast them. Teacher is comfortable using student errors as opportunities for learning.</i></p> <p>Students exhibit confidence about their ideas and share their own thinking and strategies even if they are different from others. Student ideas sometimes guide the direction of the math lesson.</p>	<p>D. Responsibility for learning</p> <p><i>Teacher encourages student responsibility for understanding the mathematical ideas of others. Teacher asks other students questions about student work and whether they agree or disagree and why.</i></p> <p>Students begin to listen to understand one another. When the teacher requests, they explain other students' ideas in their own words. Helping involves clarifying other students' ideas for themselves and others. Students imitate and model teacher's probing in pair work and in whole-class discussions.</p>
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Level 3: Teacher as co-teacher and co-learner. Teacher monitors all that occurs, still fully engaged. Teacher is ready to assist, but now in more peripheral and monitoring role (coach and assister).

A. Questioning	B. Explaining mathematical thinking	C. Source of mathematical ideas	D. Responsibility for learning
<p>Teacher expects students to ask one another questions about their work. The teacher's questions still may guide the discourse.</p> <p>Student-to-student talk is student-initiated, not dependent on the teacher. Students ask questions and listen to responses. Many questions are "Why?" questions that require justification from the person answering. Students repeat their own or other's questions until satisfied with answers.</p>	<p>Teacher follows along closely to student descriptions of their thinking, encouraging students to make their explanations more complete; may ask probing questions to make explanations more complete. Teacher stimulates students to think more deeply about strategies.</p> <p>Students describe more complete strategies; they defend and justify their answers with little prompting from the teacher. Students realize that they will be asked questions from other students when they finish, so they are motivated and careful to be thorough. Other students support with active listening.</p>	<p>Teacher allows for interruptions from students during her explanations; she lets students explain and "own" new strategies. (Teacher is still engaged and deciding what is important to continue exploring.) Teacher uses student ideas and methods as the basis for lessons or mini-extensions.</p> <p>Students interject their ideas as the teacher or other students are teaching, confident that their ideas are valued. Students spontaneously compare and contrast and build on ideas. Student ideas form part of the content of many math lessons.</p>	<p>The teacher expects students to be responsible for co-evaluation of everyone's work and thinking. She supports students as they help one another sort out misconceptions. She helps and/or follows up when needed.</p> <p>Students listen to understand, then initiate clarifying other students' work and ideas for themselves and for others during whole-class discussions as well as in small group and pair work. Students assist each other in understanding and correcting errors.</p>

on all categories. Then, the members of the teacher learning research group³ each coded 5 sessions drawn randomly from the 13 sessions. The calculated weighted agreement (Fuchs et al., 1998) was 99% for questioning, 97% for explaining, 100% for source of mathematical ideas, and 98% responsibility for learning. Together, these techniques provided support for the robustness of the framework.

RESULTS

The central result of this research is the articulation of the framework in Table 1, *Levels of the Math-Talk Learning Community: Action Trajectories for Teacher and Student*. This framework depicts growth in a math-talk learning community in two ways. First, it is made up of four developmental levels through which the case-study class moved. The movement through the levels is from a traditional mathematics classroom in Level 0 to a classroom embracing meaningful collaborative math-talk in Level 3. Level 0 in the framework represents a traditional, teacher-directed classroom. In the Level 1 classroom, the teacher in the study began to pursue student mathematical thinking, but still played the leading role in the math-talk learning community. In Level 2, the teacher began to stimulate students to take on important roles in the learning community and backed away from the central role in the math talk. In Level 3, the teacher coached and assisted her students as they took on leading roles in the math-talk learning community.

Second, the framework examines growth that occurred within each of four components from Level 0 to Level 3. The components that make up the framework are these: (a) questioning, (b) explaining mathematical thinking, (c) source of mathematical ideas, and (d) responsibility for learning. These components have been described in prior research as key features of an effective discourse community, although much of this work has examined each component separately. Overall, questioning and explaining have received the most attention. For example, Heaton (2000) describes her own attempt to reform her mathematics instruction and explains that learning to elicit student comments through questioning was a critical first step. Other researchers focus on teachers' ability to interpret and make sense of students' explanations during class (e.g., Schifter, 1996). Although also examined by relevant research, less work has been done to explore the role of students' contributions to the mathematical content of the lesson and of students' responsibility for the learning of their peers. For example, Sherin (2002a) discusses how control of the mathematical content of a lesson may shift between the teacher and the students, not only from lesson to lesson but also within a particular lesson. Furthermore, the notion of student responsibility for each other's learning in the context of a discourse community is most often explored from the perspective of whether or not students build on each others' ideas during class discussions (Sherin, Louis, & Mendez, 2000; M. S. Smith, 2000). Examining all four of these compo-

³ Members of this group at Northwestern University included Josh Britton, Corey Drake, Kim Hufferd-Ackles, Radha Kalathil, Kim Montgomery, Miriam Sherin, and Ann Wallace.

nents—both individually and together—is an important contribution of the research reported here.

For the most part, growth occurred concurrently in each of the components of the math-talk learning community in Ms. Martinez's classroom (see Hufferd-Ackles [1999] for a more extensive analysis of this issue). The means of assistance provided by Ms. Martinez in moving through these levels is discussed later in this article.

Growth in the Components of the Math-Talk Community: Development from Level 0 to Level 3

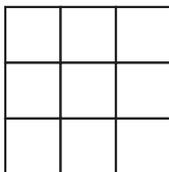
In this section of the article, we briefly explain the growth that occurred in each component of the math-talk learning community and exemplify them with excerpts of conversations from Ms. Martinez's third-grade classroom. These excerpts illustrate the learning community's path from traditional teaching to a rich and supportive learning environment.

Component (a)—Questioning

The focus of this component of the math-talk learning community is on the questioner in classroom interactions. To further children's thinking about mathematics, it is important to find out what students know and how they think about mathematical concepts. Questioning of students allows their responses to enter the classroom's discourse space to be assessed and built on by others. Questioning challenges the thinking of the person being questioned by asking for further thinking about his or her work. For this reason, questioning of students is an important part of the math-talk learning community and of reform mathematics teaching. As questioning built from Level 0 to Level 3 in Ms. Martinez's classroom, there was a shift from the teacher as the exclusive questioner to students as questioners along with the teacher. Another shift took place concurrently in the questioning component of the math-talk learning community—from a focus on *questioning to find answers* to a focus on *questioning to uncover the mathematical thinking behind the answers*.

Because the Level 0 math-talk learning community resembles the traditional, teacher-centered classroom, it is the teacher who assumes the role of question-asking, and the goal of the teacher's questions is primarily to ask students to give answers to problems. Early in the year, Ms. Martinez asked Level 0 questions that required only a brief answer, and she rarely followed up the students' responses with additional, more probing questions. Because the CMW curriculum prompted her to begin asking "Why?" and "How?" of students, Ms. Martinez quickly made the transition to Level 1 questioning. The excerpt that follows shows Ms. Martinez introducing the class to arrays for the purpose of scaffolding multiplicative understanding. Level 1 questioning is apparent in the types of questions that Ms. Martinez asked and modeled. In the excerpt below and in all excerpts that follow, the actions of the teacher and students and our commentary on what was said appear in italics within parentheses.

Level 1 Questioning: Teacher pursues student thinking.



Ms. Martinez: Now, who can tell me how many boxes of cereal I have in this container? (She points to the three-by-three array she has drawn on the board.) How many boxes of cereal do I have in this container, Carl?

Carl: Nine.

Ms. Martinez: Nine. How did you figure that out, Carl?

Carl: Because I counted them. I counted them by 3s.

Ms. Martinez: You counted them. You counted by 3s. Can you come up and show us? (The teacher is assisting the student to give a fuller explanation.)
(Carl goes to the board to illustrate by pointing to the drawing.)

Carl: I counted by 3s. There is 3 right here (row 1 of boxes). Right there (row 2). And there's 3 right here (row 3).

Ms. Martinez: So, it is like you are saying $3 + 3 + 3$. What is another way we can count? Does anyone have another way we can count? Jimmy?

Jimmy: Um, go like this. Go like this, 3, 6, 9.

Level 2 questioning is different from Level 1 because of the shift made from the teacher as the sole questioner to the students as questioners as well. This new shift in Ms. Martinez's classroom began one day when several students were working at the board. In her efforts to engage the students who had finished the problem and were waiting in their seats, Ms. Martinez told them that they each should be thinking of one question to ask the explainers when they were finished. Liz explained her work at the board for the following problem, "Ana has 3 dolls. Maria has double the amount. How many are there all together?" To Ms. Martinez's surprise, the following dialogue took place.

Level 2 Questioning: Students begin to question.

(Liz has written this labeled equation:)

$$\begin{array}{c} A \quad d \quad J \\ 3 \times 2 = 5 \\ d \end{array}$$

Ms. Martinez: Okay, Santos?

Santos: I wonder why she put the 5 in there.

Ms. Martinez: Can you ask your question to Liz? (Teacher assists student-to-student talk.)

Santos: (To Liz.) Why did you put the 5 in there?

- Liz: Because it says, "How many are there all together?"
- Saul: How come there is a "d" under the 3?
- Ms. Martinez: Can you repeat the question to Liz? (*Teacher assists student-to-student talk.*)
- Saul: (*To Liz.*) How come there is a "d" under the 3?
- Liz: Because it is for the dolls.
- Helena: es un . . plus? [Is it a plus?]
(*Liz nods in agreement.*)
- Helena: ¿Por qué pusiste el tres y el dos junto? [Why did you put the three and the two together?]
- Liz: Porque, ahí van juntos. [Because here they go together.] (*Note, "J" in Liz's work stands for "all together," juntos in Spanish.*)
- Angel: (*To Liz.*) Why did you put the 2?
- Liz: For double.

In this particular situation, Ms. Martinez could have involved herself in the discourse right away to discuss the error in Liz's solution. Instead, she waited to see if the problem in Liz's work would be clarified through the students' questioning. Students began to ask questions related to the issue of adding ($3 + 2$) rather than multiplying (3×2). Later, it took some further guidance from Ms. Martinez to resolve the issues embedded in this complex two-step problem. However, Ms. Martinez was encouraged to see the beginnings of student-to-student math talk.

At the beginning of episodes of student-to-student questioning in later classes, the teacher often prompted the questioning process with statements like "Questions for people at the board?" Initially, many of the questions that the students asked each other were modeled after questions that they had heard their teacher ask in class: for example, "What did you add?" "How did you come up with your answer?" "Can you show us on your drawing?" A positive result of this new practice was that the students in the class who were not directly involved in the discourse were actively listening to the speakers so that they did not repeat the question that another student had already asked. Sometimes students who were not outwardly participating in the questioning process gave evidence of their active listening by making comments. For example, one lower-achieving student often demonstrated active listening as he announced, "Someone already asked that."

In the following Level 3 example, students are contemplating whether one would get the same answer when adding columns of same-place-value numbers in multidigit problems from left to right or from right to left. This excerpt demonstrates the type of student-initiated questioning in order to understand one another's thinking and to understand the mathematics content that took place in the classroom when questioning reached the highest level in the framework.

Level 3 Questioning: Students initiate the questioning.

(*Ms. Martinez is in the rear of the classroom, Jamie is stationed at the blackboard. He has been called on by Ms. Martinez to share his comments about whether or not it is the same to add columns of numbers left to right or right to left with the class.*)

- Jamie:* No, because if you're taking away any numbers you gotta take away from the other ones. Are you gonna start from the right?
- Santos:* What do you mean?
- Jamie:* Right when you're taking away, yeah, subtraction, sometimes you gotta take away from the other numbers.
- Maria:* Sometimes you can start from the right *or* the left.
- Jamie:* How? Are you going to take one from the left, I mean from the right?
- Maria:* Sometimes it helps to write, like, when it's subtraction, from the right or sometimes from the left.
- Roberto:* Either way, none of the numbers are going to change. Just do the same thing you're gonna do from left to right, subtract the same thing you're gonna do from right to left.
- Jamie:* Yeah, but that's not gonna be the same answer.
- Roberto:* If you start from right to left, you're gonna subtract something and you can subtract the same thing if you go from left to right.
- Angel:* And when you go from left to right, it's gonna be the same answer.
- Ms. Martinez:* Are you still not convinced, Jamie?

(At Ms. Martinez's urging, still from the rear of the classroom, the class moves on by coming up with a problem to test. A student suggests $24 + 18$. Veronica solves the problem at the board by adding from left to right and then right to left. Veronica's work follows.)

Left to right	Right to left
$\begin{array}{r} 24 \\ + 18 \\ \hline 32 \end{array}$	$\begin{array}{r} 24 \\ + 18 \\ \hline 42 \end{array}$

- Ms. Martinez:* *(To the class.)* Okay, would either method give you the right answer?
- Class:* Yes.
- Ms. Martinez:* Yes. But we still haven't figured out what's the right answer, have we?
- Rodney:* *(He speaks from his seat.)* Veronica, where's the other tens? *(This is in reference to the additional ten created by the sum of 8 and 4 in the problem on the left.)*
- (Veronica, in response, points to the 4 in 42 in the problem on the right.)*
- Rodney:* *(He approaches the board while pointing at the 32 in the other problem.)* Where's the other ten?
- Veronica:* *(She points to the 3 in 32.)* Right here?
- Rodney:* *(He repeats.)* Where's the other ten? *(Veronica again points to the 3 in 32.)*
- Rodney:* Yeah, but eight and four equals twelve, and you just put a two right here *(pointing to the 2 in 32)*.
- Roberto:* *(He speaks from his seat.)* But you can't do it! You can't do that!
- Veronica:* Yeah, because if you put a one right here *(pointing to the chalkboard space between the numerals 3 and 2 in 32)* then it will be, uh, three hundred and twelve.

(Ms. Martinez then interjects, attempting to clarify for both Virginia and Rodney exactly what the other is saying. The discussion continues with the whole class participating. In this case, the discussion continued into the lunch period and the students asked (insisted) to continue it after recess.)

This excerpt depicts a typical instance of initiative and persistence on the part of students that was common in Level 3 situations. Several students attempted to follow, challenge, and clarify Jamie's thoughts about adding (or subtracting) from left to right. Rodney persistently pursued clarification from Veronica about her work. None of the above interactions occurred between only two people. Other students felt comfortable contributing their thoughts to the ongoing interactions. Students were no longer dependent on the teacher to initiate the process of questioning *and* to keep it going. Fewer students used the questions that the teacher had earlier modeled, but instead they focused on more specific aspects of the problem. Sometimes a student asked another student's question in different words to help the recipient understand the intent of the original question.

All students in the class asked questions, with the lower-achieving students often only mimicking what they had heard their teacher ask in a previous class or asked a simple question. However, the fact that these students were asking questions gave evidence of their comfort with being a participant in the math-talk learning community and confirmed their engagement with the discussion. By doing so, the lower-achieving students demonstrated their understanding of the general shift in the class from eliciting answers to finding out about the thinking behind the answers. At times, students seemed to ask questions because they wanted to participate, but often students earnestly wanted to pursue a specific response from the person explaining his or her work. Potential questioners were often disappointed when class ended and they did not get the opportunity to ask their question.

This excerpt also illustrates how important the teacher's role continues to be at even the highest level—Level 3. Ms. Martinez needed to intervene to clarify, to be sure that students are fully satisfied, to suggest strategies for resolving differences, and to manage time by overseeing turn taking—although much of the conversation was managed by the students.

Component (b)—Explaining Mathematical Thinking

We now turn to the second component of the math-talk learning community: explaining mathematical thinking. Although this component is closely connected to the process of questioning, here we attempt to focus exclusively on the process of explaining as we go through each level of the math-talk learning framework individually.

As students in Ms. Martinez's math-talk learning community became more comfortable and more able as explainers (and as Ms. Martinez began to facilitate this development), the community moved from Level 0 to Level 3 in this

component. Significant support for the growth in this area was found in the social climate that the class together developed that effectively supported student explainers. Initially, standing in front of one's peers to communicate mathematical ideas was a daunting task for many students, especially shy students. Many chose to sit back down in their seats after writing work on the board rather than to accept the challenge of staying in front of the class and talking. However, as the math-talk learning community developed, students' attempts at explaining were scaffolded by supportive classroom colleagues. This support allowed the development of explaining to progress. As students learned to explain their own mathematical thinking more fully and fluidly, they made significant contributions that could then be questioned or built on by other students and assessed by the teacher.

In the Level 0 math-talk learning community, students often gave answers to Ms. Martinez's questions in one to a few words. Student explanations consisted of short interchanges between the teacher and the students. Questions asked of students by the teacher were primarily answer-focused. At times the teacher did not even wait for an answer from the students and gave it herself. The following interaction illustrates a Level 0 explanation of mathematical thinking, showing that students' explanations of their work were focused on providing the correct answers. The teacher was not looking for more explicit strategies or thinking from the students and, not surprisingly, they did not offer it. Students were solving this problem: "Joey bought 5 packs of gum at the store yesterday. Each pack has 7 sticks of gum. How many sticks of gum does he have?" In the following excerpt, Ms. Martinez provided a fuller explanation for Charlotte herself rather than have Charlotte explain.

Level 0 Explaining mathematical thinking: Teacher assistance focuses only on correctness of answers.

(Charlotte has drawn the following on the board:)

///// ///// ///// /////

Ms. Martinez: Okay, so how many packs has Joey bought?

Charlotte: Five.

Ms. Martinez: Five packs, so we have to draw a fifth pack, right? Draw it.

(Charlotte begins drawing a fifth pack on the board. After finishing her drawing and providing an answer, Charlotte goes back to her seat.)

///// ///// ///// ///// ///// = 25

Ms. Martinez: Okay, here's what Charlotte did. She has five times seven equals twenty-five. So she has (*pointing to each part of Charlotte's drawing at the black-*

board) one, two, three, four, five packs. Is there something that is not exactly right in here that you might have missed?

Charlotte: The answer?

Ms. Martinez: Okay what, what is wrong with the answer? First, look at the picture, okay? Is there something wrong with the picture? Are you missing something?

Charlotte: Need more sticks.

Ms. Martinez: Excuse me?

Charlotte: Need more sticks. There's only five sticks in each.

Ms. Martinez: You have (*pointing to the drawing*) one, two, three, four, five sticks. And how many is it supposed to be, Charlotte?

Charlotte: Seven.

Ms. Martinez: Seven. Because you have seven sticks of gum. Okay, that's where you went wrong, because there's seven sticks of gum. Each pack, in each pack, you need seven. (*Ms. Martinez adds the extra two sticks for each row in the drawing.*) Now she has a pack of seven sticks. Now we have five packs of seven sticks. Now, Charlotte, what is your new answer?

$$\begin{array}{cccccc} \text{//} & \text{//} & \text{//} & \text{//} & \text{//} & = 35 \\ \text{//} & \text{//} & \text{//} & \text{//} & \text{//} & \end{array}$$

Charlotte: (*She pauses.*) 35?

Ms. Martinez: 35, right. We have seven plus seven plus seven plus seven plus seven, which all together equals 35. Let's go on to the next problem.

In this excerpt, Ms. Martinez prompted Charlotte to correct her drawing rather than explore the reasons behind her choice of a solution. In other words, Ms. Martinez's goal was to direct Charlotte to the correct answer rather than to understand her thinking and the reason for her error. She never found out why Charlotte's original drawing started with 5 sticks per pack. In addition, Ms. Martinez described Charlotte's work to the class rather than have Charlotte explain her thinking. Had Charlotte been given the opportunity and had she possessed the capacity to explain her work, Ms. Martinez would have had more opportunity to understand her thinking process. For instance, after Charlotte commented that there were only 5 sticks (per group), Ms. Martinez did not ask her to explain herself more fully. Eventually, Ms. Martinez told the class, "Seven. Because you have seven sticks of gum." Finally, Ms. Martinez abruptly left Charlotte's problem and moved on to the next student's work without having knowledge of any of the other students' understanding of the corrected solution. In summary, this excerpt illustrates the lack of attention to student thinking in Level 0 of the developmental trajectory of explaining. This situation shows that Level 0 explaining is the counterpart to Level 0 questioning because both the questioner and explainer are focused on answers only.

Ms. Martinez made the transition to fuller student explanations of mathematical thinking in her classroom by beginning to probe students more deeply. Level 1 explanations were given as students shared information about their thinking in response to the teacher's probing. The first attempts at fuller explanations were labo-

rious for students because they were uncomfortable responding to several consecutive questions while standing at the front of the room. Many students preferred walking over to the teacher (who was often standing nearby) and talking to her privately. The following example shows students explaining their work after solving the word problem at the board, “Carrie is playing with 6 girls. How many fingers are there in the group?”

Level 1 Explaining mathematical thinking: Teacher assists students in their brief initial attempts.

Ms. Martinez: Explain what you did.

Saul: 6 times 7.

Ms. Martinez: Why did you write a 6?

Saul: 6 girls.

Ms. Martinez: Why did you write a 7?

Ms. Martinez: (*She waits a moment.*) You can't explain? (*Saul shakes his head “no.”*)

Ms. Martinez: Okay, have a seat.

Ms. Martinez: Henry, can you explain to the class why you put 6×5 ?

Henry: There are 6 girls (*pauses*) multiplied by 5. You get 30.

Ms. Martinez: Can you say it again to the class, loudly?

Henry: (*Inaudible.*)

Student in back: I can't hear. Can you say it louder?

Ms. Martinez: Henry, you have to face the audience.

Henry: 6 girls multiplied by 5 (*pause*).

Ms. Martinez: Since Henry's voice is quiet, I will repeat it for him. 6 girls counted by 5 equals 30.

Ms. Martinez: Where did you get the 5?

Henry: Because that makes 30.

Ms. Martinez: Okay, you can sit down.

This excerpt indicates that facilitating students' explaining of their thinking required patience on the part of the teacher; there were many long pauses as students considered what to say. It would have been much quicker for Ms. Martinez to show students how to arrive at an answer of 30. Furthermore, taking on the central role of explainer in the classroom discourse was uncomfortable for many students, as Saul and Henry illustrate. They were familiar with the conventional expectation that they say only a word or two and then sit down; they were not accustomed to identifying and explaining their own thinking processes.

The Level 2 explanation of mathematical thinking began after students became more comfortable with the process of communicating about such thinking. At this level, the students still required probing and some assistance in clarifying their thoughts from Ms. Martinez and, increasingly, from other students. Student explainers grew more confident that their thinking was valuable, and they became less shy about telling their mathematical ideas. They grew to expect that providing a numerical answer was not enough information. Furthermore, the classroom social norms grew to embrace and encourage student speakers. Students began to

listen to one another more actively, help from other students was accepted as positive, and often students applauded after their classmates gave explanations of their work at the board. Thus, being the center of classroom discourse became less scary for students, and more students volunteered for the opportunity to tell their thoughts about the mathematics. The following excerpt demonstrates a Level 2 explanation of mathematical thinking. In this excerpt, Ms. Martinez asked Santos to explain his multidigit addition work on the board.

Level 2 Explaining mathematical thinking: Teacher assists students as they provide fuller, more comfortable explanations.

(Santos has written his work on the board:)

$$\begin{array}{r} 1 \ 1 \\ 2 \ 5 \ 8 \\ + 3 \ 7 \ 4 \\ \hline 6 \ 3 \ 2 \end{array}$$

Ms. Martinez: Santos, do you think you can explain this?

Santos: *(He stands next to his work at the board.)* Eight plus four is the two, and then the ten goes over here, over the five. That equals a hundred and thirty, the hundred goes here, over the two. You end with six hundred thirty two.

Ms. Martinez: Has he explained everything?

Class: No.

Ms. Martinez: He still hasn't explained what the ones are doing up there, has he?

Santos: Oh, well, without the ones it would be a different answer.

Ms. Martinez: What do you mean?

Santos: Without the ones, it would be five hundred and *(pause)* . . . no, yeah, five hundred and twenty-two.

Ms. Martinez: All right, but, how do you know that is the right answer and that your other answer isn't?

Santos: Because, I know how to count.

Ms. Martinez: You know how to count what?

Santos: I know how to carry. I know that you need to carry here to get the right answer.

Ms. Martinez: You know how to carry what?

Santos: I know how to carry the ones. The numbers, I know how to carry these numbers to get the answer.

Ms. Martinez: What does that mean, carry the ones?

Santos: That, you put the ones up here, on top of these, the tens and the hundreds.

Ms. Martinez: Why?

Santos: Because, it needs to be up there. The two is for twelve, and you put the one up there.

Ms. Martinez: Why?

Santos: Because if you don't put the ones, it'll be one thousand, five hundred . . . and that would be the wrong answer!

In this example, Santos demonstrated a greater comfort level at the board than Henry did in the Level 1 example. He was not surprised by or uncomfortable responding to Ms. Martinez's probing. Santos knew that he had not described all of the steps in his work. Rather than jump in herself and explain the process of making a new ten or hundred in more "correct" mathematical language, Ms. Martinez allowed Santos to explain in his own words. Santos illustrated an aspect that was recurring more frequently as the students moved to Level 2 explaining. He confidently staked a claim and continued to defend the claim using his own words. Ms. Martinez's questioning helped Santos's explaining to be more complete. Listening students could follow along more easily, and they remained more engaged. In this particular excerpt, the teacher was probing the student's thinking. In other cases at Level 2, students acted as questioners as well. The teacher's role in assisting the explaining component continued to be very important. Here, Santos needed further assistance from the teacher or from a student to explain that the ones were 1 ten and 1 hundred.

In the third level of explaining mathematical thinking, students began to defend and justify their mathematical ideas more confidently and thoroughly. Although Ms. Martinez was ready and available to probe and guide students in making their explanations more complete, their responses became more extensive and thorough and needed less assistance. During this classroom segment, students had worked together in pairs or groups to solve multidigit addition problems. The group made up of Veronica and Lou put the following work on the board and then they took their seats.

Level 3 Explaining mathematical thinking: Students engaged in full, confident explaining without overt assistance.

$$\begin{array}{r} 438 \\ +271 \\ \hline 600 \\ 100 \\ 9 \\ \hline 709 \end{array}$$

- Ms. Martinez:* (She directs her question to the class.) Questions for this group?
- Maria:* (She asks Lou.) Can you show us how you're adding?
(Veronica steps to the board next to the work she and Lou completed.)
- Veronica:* (She responds to Maria.) You mean all of this (motioning to their work underneath the original written problem)?
- Maria:* Yeah.
(Veronica turns to Lou, who has gotten her attention; he wants to be the one to answer).
- Lou:* (He comes to the board, and begins his step-by-step explanation.) I added the hundreds, the four and the two together, and I got six hundred. (Veronica cuts him off and steps to the board to speak to him.)

- Veronica:* No, just this. (*She points to the work just above the final answer and whispers to Lou. She then sits back down. She is directing him to explain just the final adding step, not all of the adding of the places.*)
- Lou:* (*He points to the final hundreds place, in the answer.*) This was six hundred, and then another hundred, so that was seven hundred, and so with the ones, seven hundred and nine.
- Jamie:* How did he get the seven hundred? How did you get the extra hundred?
- Veronica:* He just said it. He just said it. He said he added, and he got the hundred.
- Lou:* (*He is now speaking from his seat.*) I made it with the seventy, and the thirty, which gave another hundred.

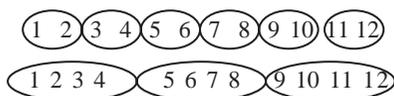
In this excerpt, Lou capably described the mathematical thinking that he used to solve the multidigit addition problem from left to right. He repeated the steps in his thinking process without the probing of Ms. Martinez and despite the interruptions by Veronica (Lou's slightly overbearing partner). Lou also took ownership of the explanation process and answered questions about it even after he finished telling about his group's approach to solving the problem and was sitting down. In Level 3 of the math-talk learning community, important information came from student discussion as well as from the teacher.

Component (c)—Source of Mathematical Ideas

At Level 0, the teacher presented mathematics content by standing at the board and telling students how to solve problems in a procedural manner. Students watched so that they might imitate the teacher, and then they did much of their work individually. Initially, she focused on having students copy word problems word for word from the board and solve them individually. She went from table to table and told students how to do the problems, sometimes by doing the problem for them on their papers. Students watched in order to imitate her procedures.

Ms. Martinez's class shifted to Level 1 when Ms. Martinez began to elicit student ideas as she presented content. This shift was facilitated by the conceptual focus of the curriculum. Eliciting students' ideas allowed her to uncover their previous knowledge and current misconceptions and to follow their developing understanding about the material. Student input allowed her to modify the course of lessons according to the evolving ideas of the students. The classroom excerpt below typifies Level 1 sources of contributions to teaching and learning of mathematics content. It shows Ms. Martinez standing at the board in the front of the room and beginning to modify the pace of her lesson (comparing multiplication by twos and fours) to the students' understandings. All students were sitting in their seats, and the majority appeared engaged. The class relied on the drawing below in their discussion. It showed that in the two's "count-by," there are six sets of twos between the numbers 1 and 12. In contrast, the four's "count-by" yields three sets of fours in the same number range.

Level 1 Source of mathematical ideas: Teacher begins to use student thinking as part of the mathematics content.



Ms. Martinez: Now, for the two's count-by you go two-four-six-eight-ten-twelve. And with the four's count-by you go four-eight-twelve. So, six fingers for the two's and three fingers for the four's. What is this three in comparison to six? If the third finger for the two's is six, what about the third finger for the four's? Charlotte?

Charlotte: Twelve?

Ms. Martinez: Twelve, good. Do we notice anything between the six and the twelve? Michael? This six right here. (*She points to the six in the number set for the two's count-by.*) And this twelve right here? (*She points to the twelve in the number set for the four's count-by.*)

Michael: Um, it'd be like times the . . . no, double the six would be twelve!

Ms. Martinez: Good. You double the six it'll be twelve. Any other number patterns that you see, Maria?

Maria: Six plus six is twelve?

Ms. Martinez: Okay, that's what Michael had said. If you double the six, it'll be twelve. Anything else about any of the numbers up here, anything that we see repeatedly? Liz?

Liz: Twenty.

Ms. Martinez: The twenty? Okay, what finger is the twenty on, Carrie? In the two's count-by, what finger?

Carrie: On the ten.

Ms. Martinez: On the ten, good. How about in the four's count-by, what finger is the twenty on?

Carrie: The fifth.

Ms. Martinez: So you used only five fingers to get to twenty in the four's count-by but you used your entire fingers to get to twenty in the two's count-by. So the twos are only taking up two numbers, which is why you use so many fingers. But the four, the four is taking up four numbers per finger, you use two numbers for one finger for the twos.

In this excerpt, students demonstrated more involvement with the lesson than they did when Ms. Martinez used the Level 0 tactics of telling students how to do mathematics. Students began to try to think about and understand the mathematics rather than merely attempt to imitate the teacher's words and actions. Thus, teaching in Ms. Martinez's room shifted from a procedural focus to one in which students were searching for meaning as the class moved from Level 0 to Level 1 in this component of the math-talk learning community.

As was evident in other Level 2 components, Ms. Martinez began to shift her physical presence to the side or to the back of the room at this point in the trajectory. It is important to note that Ms. Martinez began to allow more opportunities for students to explore content and suggest alternative and multiple methods. She

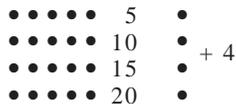
did this by asking more open-ended rather than answer-driven questions. Ms. Martinez also continued to ask for other students' strategies, even after a correct one had already been presented. In doing this, Ms. Martinez demonstrated her willingness to learn the alternative strategies herself. At times, she asked students to explain their strategies more than once so that she could fully understand them. As she took on the role of co-learner in the classroom, she modeled aspects of learning from others that students later mimicked (e.g., how to ask questions to support understanding). Hearing multiple strategies allowed Ms. Martinez to assess the understanding and possible misconceptions that students held as they moved through each content domain. The following excerpt is from a class in which Ms. Martinez gave students opportunities to solve array word problems. We summarize the students' methods here rather than give the full transcript in the interest of space.

Level 2 Source of mathematical ideas: Student methods form much of the content.

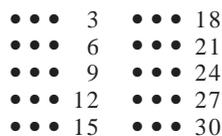
(Santos made up the problem, "In my garden I had 4 rows and 6 columns of lettuce heads. How many lettuce heads did I have?" He drew the following picture on the board.)



In response, students offered a number of solving strategies. Angel said that you could count each lettuce head. Nick said that you could count by fours and showed how he would do that by using the vertical groupings, "4, 8, 12, 16, 20, 24." Roberto said that he would count by fives (using the horizontal groupings) and then add four (the vertical grouping left over), as shown in the diagram below.



Jimmy solved the array, "There are 6 in each row, 6 and 6 is 12, the others are 12, I added 12 and 12." After Ms. Martinez added another row to the problem, Maria counted by threes to find her solution:



Henry said he counted by tens to solve the five-by-six array problem. Ms. Martinez responded with "How can I use tens to get my answer?" Henry then

showed the class how he grouped two vertical columns to make ten. There were three groups of ten. Jessie counted by twos to thirty using vertical groupings as Maria did earlier to count by threes. Ms. Martinez asked several of the students to explain their thinking twice, which allowed her and the other students opportunities to understand the method more fully.

At Level 2, Ms. Martinez also became adept at using students' strategies that contained errors for opportunities to learn. This can be observed in the excerpt used earlier to illustrate Level 2 in the questioning trajectory. The answer for the word problem, "Ana has 3 dolls. Maria has double the amount. How many are there all together?" contained an error ($3 \times 2 = 5$). Ms. Martinez then stimulated the class to think carefully about the language of the problem to allow the students to uncover the error. The class discussion began to focus on the words *double* and *all together* and how their meanings affect the processes of problem solving in this situation.

Reaching Level 3 of the sources of mathematical ideas trajectory depended on two factors. First, students gained confidence that their ideas about mathematics were valid and important. Second, Ms. Martinez became convinced that the ideas students contributed *were* important to explore. Ms. Martinez articulated the latter in this way: "I think the kids explain it in a language that kids can understand." Therefore, she gave students discourse space when they wanted to volunteer their thoughts. At times, she would stop what she was doing and allow a student to take the chalk and explain their idea at the board. Sometimes students would expound upon a strategy that was explained in the curriculum that Ms. Martinez had not yet introduced to the class. She often recognized the importance of these student-initiated strategies after reading the curriculum and then quickly integrated them into the class discussion. For example, the following excerpt comes from a mathematics class in which Ms. Martinez was showing students a way to multiply by sixes, which builds on their knowledge of multiplying by fives. Earlier in the class, Roberto had already verbalized that this is the method he used to solve 6×7 .

Level 3 Source of mathematical ideas: Student strategies are built on as mini-lessons.

Ms. Martinez: I am going to show you a different way to count by 6s, similar to how Roberto said.

Ms. Martinez: How many groups of 7 are there in 6×7 ?

Students: 6.

Ms. Martinez writes this on the board:

$$6 \times 7 = 7 + 7 + 7 + 7 + 7 + 7$$

$$35 \qquad 35 + 7 = 42$$

Ms. Martinez: It is easier to multiply by 5 isn't it?

(Ms. Martinez is in the middle of explaining this strategy, and Jimmy intervenes.)

Jimmy: I have another way.

Ms. Martinez: Okay.

Jimmy: Two of them equals 14, another two is 14, and another two is 14.

Ms. Martinez: Good, can you come up and show us? (*She hands Jimmy the chalk.*)

(*Jimmy writes on the board:*)

$$\begin{array}{ccc} 7 + 7 & 7 + 7 & 7 + 7 \\ 14 & 14 & 14 \end{array}$$

(*Jimmy tries to explain how he gets 14 and 14 and 14 to add up to 42. 4 and 4 and 4 and 30. . . 14 and 14 is 28. . . he stumbles a bit and uses his fingers to show $28 + 14 = 42$.*)

Ms. Martinez: Good explaining, he didn't give up even though he was a little tongue tied.

Ms. Martinez: What Jimmy explained here is kind of like what Chris was explaining for counting by 2s.

Ms. Martinez: Okay, solve this problem in your journal. Use the way that Jimmy came up with or the way that I showed you with Roberto's help.

(*She writes on the board: $6 \times 9 =$*)

Both the doubling strategy used by Jimmy and the building on fives facts strategy initially introduced by Roberto were part of the curriculum lesson for this class. Instead of teaching these strategies in a traditional "telling" manner, Ms. Martinez allowed them to emerge from the students. Then she followed up to clarify and relate them. Teaching in this way, the class still explored the target mathematics content, but because of their contributions it was covered in a way that effectively engaged students.

Component (d)—Responsibility for Learning

As the math-talk learning community developed, responsibility for learning shifted as students became increasingly invested in their own and one another's learning of mathematics. Students began the year in Ms. Martinez's room only as passive listeners as their teacher led the class in a traditional manner. When student thinking began to be elicited, students became more engaged and involved in classroom discourse as speakers and listeners. Their responsibility for their own learning was indicated by their desire to ask questions in class, their eagerness to go to the board to demonstrate their understanding of problems, and their volunteering to engage in the work of and to assist struggling students at the board. Students grew to expect that their mathematics contributions would be positively received by the teacher and by other students. Having students' ideas in the classroom discourse space enabled students to help each other. A respect and concern for the learning of others became a by-product of Ms. Martinez's students actively engaging themselves in their own learning.

Teaching in a reform-oriented mathematics classroom is a challenging task. For students to see themselves as co-learners and co-teachers in the classroom was a substantial help to Ms. Martinez because all of the students began to see themselves

as responsible, in part, for the learning of everyone in the room. One student demonstrated this aspect of the math-talk learning community after he explained his strategy for solving his 10-by-12 array at the board. He carefully explained his steps and then earnestly asked, "Teacher, do you understand?" Ms. Martinez graciously responded, "I understand completely." Chris's question illustrated that the students grew to be confident about their mathematical thinking and that they wanted the expression of their thinking to be meaningful to others, including their teacher.

From early in the school year, Ms. Martinez desired to engage all of her students in her teaching of mathematics. When the class was at Level 0 with respect to student responsibility, Ms. Martinez repeated student responses originally directed to her so that all the students in the class could hear. Students passively listened to redirected statements of their peers and did not engage in the thinking of other students. In other words, students' fundamental belief was that they needed to listen to and imitate the teacher (not other students) in order to successfully learn mathematics. In the Level 0 classroom, students did not demonstrate confidence in the ways that they solved problems. Ms. Martinez unilaterally verified the correctness or problems in student work. Students did not have the opportunity to develop a full understanding of the mathematics involved because the focus was on fixing work so the student would get the correct answer. Students were uncomfortable being in the front of the room and unaccustomed to discussing the ways in which they found their answers. Students' responses were quietly directed to the teacher and not intended for the whole class to hear. Under these conditions, Ms. Martinez assumed the role of explaining, which meant also choosing the language that would convey the initial student ideas to the class.

Ms. Martinez's class moved to Level 1 in the responsibility for learning component as she began holding her students accountable for listening to one another and as she began to focus on thinking and not just on answers in the evaluation of student work. Her explicit tactics that were intended to stimulate accountability led students to believe they should listen to what was being said by other students because they might be called upon to repeat something that was said in the course of the discussion. Ms. Martinez made an effort to ask about student thinking, but not to repeat for the students herself. She let other students start taking on this role. Although students became able to repeat what other students were saying, this did not seem to advance the class discussion. However, it did often succeed in keeping students alert in class and honed listening skills. At times, Ms. Martinez had students repeat correct and incorrect information without also making decisions or comments about the validity of the information. Although this was a move forward in terms of holding students accountable to listening to one another, the repeating process often impeded the flow of the class. As Ms. Martinez stopped to have students repeat, the continuity of the potentially meaningful discussion was often lost.

In an effort to build accountability and scaffold students into taking responsibility for their learning, Ms. Martinez abandoned this rather limiting (but perhaps a helpful transitional) practice for one that was more successful in engaging

students to think about mathematics. The class shifted to Level 2 in student responsibility for learning when Ms. Martinez began having students explain the mathematical thoughts of others more fully and in *their own words*. This resulted in student listeners comparing the work of others with their own thoughts. Students were challenged to spend time trying to understand what others meant in their explanations instead of mindlessly reiterating the words used by them. At Level 2, Ms. Martinez facilitated deeper student thinking and responsibility by asking substantive questions, such as “What would you have done, Nathan? Would you have counted the same way he did?” and “What was the difference between how Michael counted and how Nathan counted?” This process required students to think more deeply about their own *and* another students’ ideas. This reworking of explanations eventually helped even lower-achieving students to compare strategies. Ms. Martinez also modeled for students the questions that helped them participate in the evaluation process. By being able to decide whether or not they agreed or disagreed with the explainers, they were able to shift into the roles of critic, helper, and supporter with respect to other students’ work.

The shift to Level 3 in this component of the Levels of Math-Talk Learning Community framework occurred as students took the initiative to clarify other students’ work and ideas for themselves and for others during whole-class discussions and small-group interactions. The teacher alone did not give the constructive feedback for student work. Rather, it was co-evaluated by all of the participants in the math-talk learning community as part of the ongoing supportive helping process.

At Level 3, Ms. Martinez was able to have one or more students help another student while the rest of the class moved to another explanation. Thus, Ms. Martinez was able to focus on more students in the span of the classroom time while students evaluated and helped each other make corrections in their work. Ms. Martinez reported that refraining from making verbal assessments of student work when students could be making those comments was a challenging change, but she thought it had been very beneficial. Individual students also took responsibility in the Level 3 classroom by initiating group practice such as with the count-bys (e.g., count-by 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60) during slow parts of class periods or during times when the teacher was helping individual students.

The following example of Level 3 responsibility for learning shows one student’s quest for place-value understanding. Several other students became involved in assessing and assisting her understanding. Ms. Martinez acted in a supporting rather than central role in this situation.

Level 3 Responsibility for learning: The whole class acts as teachers when students do not understand—students assist other students in understanding.

(Henry has solved this problem at the board from left to right.)

$$\begin{array}{r} 485 \\ + 376 \\ \hline 700 \\ 150 \\ \hline 11 \\ \hline 861 \end{array}$$

Ms. Martinez: Liz, do you have a comment?

Liz: How come he has a one over here, one in the ten and the other in the ones if there are 11 ones?

Maria: I know why, because 6 and 5 is 11 and he can't put that in one column.

Liz: (She goes to the board.) How come they put 1 in the tens and 1 in the ones, how come one is over here and one is over here?

Maria: $6 + 5$ is 11.

Ms. Martinez: Does someone else want to try to explain?

(Six students raise their hands to respond.)

Rodney: If we put the whole thing here it would all be ones, but this is tens and ones.

Ms. Martinez: How about it Liz, understand? Satisfied? (Pause.) She is still a little unsatisfied. Who can try to explain?

Helena: The other time you said. . . [to add left to right] we can count first the 100s, then the 10s and then the ones, we have 11 here, we are counting ones not 10s.

(Eight students raise their hands to try to explain.)

Chris: Because 11 has one ten.

Ms. Martinez: And?

Chris: Because 11 has one ten and you can't put 11 in the ones column.

Ms. Martinez: Why?

Chris: Because it goes in the ten column.

Ms. Martinez: But one is in the ones.

Liz: The 11s are the ones, and you put the tens and the ones.

Santos: Teacher, I think I know.

Maria: (She has approached the board.) $6 + 5$ is 11.

Ms. Martinez: You keep saying the same thing, but what does it mean? One more person.

Saul: (He has now joined Maria at the board.) 11 has one ten, so it goes here (points to the tens column).

Santos: If you put it in the bottom, it would be 862.

Students: No, 8000 (meaning it would be 8611, moving 8 left to the thousands place).

Ms. Martinez: Liz still has a question.

Liz (strongly): THE 11s ARE STILL THE ONES, HOW CAN YOU HAVE ONE IN THE TENS AND ANOTHER IN THE ONES?? (She does not see eleven as 1 ten and 1 one or cannot use that knowledge here.)

Ms. Martinez: Saul, do you get it?

Saul: It would be 8611. (He writes this in the answer line of Henry's problem.)

Ms. Martinez: We don't have any more time.

Students: Aaahhhhh!

Ms. Martinez: We will have to think about the best explanation for tomorrow, think about it tonight.

Liz continued to search for understanding with the students around her at the lunch table following this class. As a result of further student interactions, Liz told Ms. Martinez that she was satisfied with her understanding when the class returned from lunch.

This excerpt illustrates Ms. Martinez having chosen to involve herself as a supporter of the discussion while allowing students to take the central explaining role. Rather than resolve Liz's misconceptions herself, she gave other students opportunities to try to understand Liz's thinking and to help her by explaining in such a way that Liz would understand. Students were so engaged in the Level 3 situation that they impatiently waited to contribute to the discourse. Many clamored for the opportunity to help Liz understand the situation. Students were visibly disappointed when they had to stop interacting around this mathematical dilemma and go to lunch. Liz's press for understanding and Saul's explaining at the board illustrate the progress made by the shy students in this classroom. These particular students were initially very reluctant to share their thinking. They grew confident and comfortable enough to initiate sharing their thoughts. Liz even continued to pursue understanding in the face of many students who did not seem to share her perspective, but were instead trying to fix it.

As students learned to listen in order to understand each other's thinking in the Level 3 classroom, several positive classroom consequences resulted. For example, when a number of different solution strategies were possible for problems or situations, students listened carefully to contributions that others made to the discussion to be sure that what they would contribute would be new information. Listening to understand also launched students in the collaborative initiative to become assisters for one another, as in the excerpt above. To successfully assist one another, they needed an awareness of their own understanding of the material and they needed to understand one another's perspectives. Ms. Martinez increasingly relied on students who understood material to assist her in teaching students who did not yet fully understand. Students offered help and accepted help graciously from their fellow co-learners.

Moving Through the Levels

The case-study class moved quickly from Level 0 to Level 1 in all components of the math-talk learning framework. This movement can be attributed in part to the use of the CMW curriculum that supports a focus on student thinking and explaining of ideas. Changes similar to those in Level 1 were observed in Everyday Math classrooms (Mills, 1996; Mills, Fuson, & Wolfe, 1999).

Ms. Martinez's class then spent approximately 8 weeks at Level 1 before moving to Level 2. Because the Level 1 to Level 2 transition represents the greatest shift

in the classroom—from the teacher as the central figure in the math-talk community to the students as central figures—this transition may represent a difficult change for the teacher and class to make and may take time even with students who are not also learning English.

The class operated as a Level 2 math-talk learning community for 3 months before exhibiting a majority of Level 3 characteristics. This transition to Level 3 was even more gradual than the Level 1 to Level 2 transition. Examples of Level 3 attributes occurred more frequently over time as students took on more central roles in the math-talk learning community. The class began to function fully as a Level 3 community early in March. Ms. Martinez left the school for her maternity leave early in April.

There were fluctuations from the overall upward trajectory in levels whenever new topics were introduced. Students needed to learn the new vocabulary and concepts of a new topic in order to function as a higher-level math-talk community. These drops in level were particularly apparent when the class shifted in December from their extensive work on multiplication and division to multidigit addition and subtraction.

It sometimes took the students several days to begin to resume their roles as question askers and explainers as they learned the language and representations of each new domain. During this adjustment time, Ms. Martinez functioned in a more central position and was responsible for more of the discourse. She stated, “Once I go over it and give them a sample of how I would explain, they seem to catch on better. They are more sure of what words to use and what drawings they can use, even though I tell them that whatever drawing is fine. But they are not sure if what they are going to say is right.” Ms. Martinez asked many Level 1-type questions to support student familiarity with the language in the new areas of mathematics. Although Ms. Martinez resumed a more central role in these classes than she had in the preceding weeks, her goal was to familiarize students with the language of the new domain (e.g., place value) so they could resume their more significant roles in the math-talk learning community. Her growing belief in the abilities of her students motivated her to support them to participate in the math discourse quickly in each new domain. Furthermore, because the expectations of the class as a whole had changed, she noted, “When it’s more of a teacher-centered class, I tend to lose kids.” After brief functioning at Level 1, the class returned to higher Level 2 and Level 3 characteristics as they explored the new mathematics together.

Students’ must have a grasp of the language of the domain of mathematics in order to carry on math talk both to describe one’s own thinking question or extend the work of others. M. S. Smith (2000) similarly found that the teacher in her study was most directive at the beginning of a unit, and that later the class as a whole was more comfortable discussing the content. Mendez (1998) stated a similar conclusion in her study of robust mathematical discussions, “The need for significant mathematics within the students’ zones of proximal development was found as another necessary condition for robustness” (p. 146). In other words, the mathematics must be accessible to students or familiar enough for them to be able to

participate in meaningful discourse. As teachers move through the year, they will need to fall back to Level 1 or Level 2 to assist students in building vocabulary and concepts in new content areas. Furthermore, not every day includes extensive math talk. Some days may involve individual or student-assisting paired practice.

Teacher Actions Facilitating Transitions to New Levels

Ms. Martinez enacted particular actions to support class transitions from level to level across all of the components of the math-talk learning community, as shown in the summary in Table 2. Each of these teacher actions was followed by a corresponding change in student actions. To move from Level 0 to Level 1, Ms. Martinez began to focus more on students' mathematical thinking as they arrived at answers and less on the answers themselves. To move from Level 1 to Level 2, Ms. Martinez increasingly expected students to take on substantial roles in the math-talk learning community, and she assisted them in learning these roles. Moving from Level 2 to Level 3 involved increasing expectations on Ms. Martinez's part that students would take central roles in the math-talk learning community; she gave them the space that they needed to take ownership of the roles, then she coached and assisted them as they became major participants in the math talk.

Table 2

Ms. Martinez's Means of Assistance for Making the Transition to a New Level

	Means of Assistance
Level 0 to Level 1	Ms. Martinez began asking questions that focused on mathematical thinking rather than answers. She assisted students when they attempted this new task by modeling language.
Level 1 to Level 2	Ms. Martinez began to fade from the central role in the physical and discourse space and assisted students in taking on substantial roles in the discourse community. She probed for student thinking and assisted students in clarifying their thoughts when necessary. She asked questions that were open-ended rather than "fill in the blank" and sought extended descriptions of multiple student strategies.
Level 2 to Level 3	Ms. Martinez expected students to take on central roles and gave them the physical and discourse space to do so. She coached and assisted students in their participatory roles in the discourse. She expected students to assist one another voluntarily and assisted them in doing so. Ms. Martinez actively monitored interactions and remained available from the side or back of the room to assist when students needed clarification or when an interaction needed support.

Strength of the Student Community

The continuity of the community in Levels 2 and 3 was not totally dependent on the presence of Ms. Martinez. For example, continuity was apparent even when there was a substitute teacher during one observation near the end of November.

In this class, individual students wrote solution strategies at the board. The substitute teacher asked students one by one to explain their work and tried to move quickly through the problems. Instead, her pace was interrupted numerous times by students saying they were not ready to move to the next problem because they had questions or comments for the student explainers. The substitute was amazed by the students' initiative. Similar events occurred after Ms. Martinez left for her maternity leave in April. Her replacement was unaccustomed to the level of involvement that the students initiated. This new teacher had to work to increase his level of expectation for the students and his own role in the mathematics classroom to fit into the existing math-talk learning community.

CONCLUSIONS AND FUTURE RESEARCH

Principles and Standards for School Mathematics (NCTM, 2000) emphasizes the importance of learning in a mathematics community because it fosters students' communication of mathematical ideas and helps students to build mathematical understandings. Discussion of mathematical ideas provides opportunities for students to reason, defend, and prove their conceptions to one another. Over the course of the year, Ms. Martinez's students reached these challenging communication standards. Developing an environment where this type of math talk takes place can be a daunting task for teachers. By specifying components and levels in the creation of such an environment and by describing specific means of assistance that Ms. Martinez and her students provided to each other, this article offers assistance to others trying to build such a community. The framework can guide teachers to listen to their students, to draw out students' ideas, and to encourage students to listen to each other. Moreover, this study demonstrates that an effective math-talk learning community *can* be developed in urban classrooms, even with students still learning English. For this reason, we believe that the results described here are widely generalizable.

This research resonates with, and extends prior research on, the development of mathematical discourse. Like previous research, we have argued that opening up one's classroom to students' ideas is the critical first step in achieving a discourse community (e.g., Fennema et al., 1996). However, this article examines the steps beyond the initial Level 1 community and describes the interrelated components in a Level 2 and Level 3 discourse community.

Since its development, the math-talk framework that resulted from this study has been used with over 200 preservice and in-service teachers across multiple school settings in professional development situations. Teachers expressed the belief that the framework is accessible to them and also doable; it provided them with a vision for change. Specifically, many teachers attributed changes in their practice to conversations about the framework held in after-school mathematics meetings. The math-talk framework is one element that is possibly useful in scaffolding teacher change. Although this study focused on the change in practice of a person relatively new to the teaching profession, we have also found similar changes among more

experienced teachers who discussed the framework described in this article (e.g., Drake, 2000).

Future research needs to focus more specifically on what happens during the transitions between levels in each of the components and how those transitions could be effectively supported in classrooms. It also needs to examine various ways to assist teachers in making these changes. In this research, the teacher was assisted by the research-based CMW curriculum, the reform-focused school administrators, and weekly feedback from the researcher. These together facilitated rapid change that could be studied and described over a several-month period. For widespread impact, there is a need to understand how to assist thousands of teachers in their movement through the levels of math-talk learning community with individual weekly support. Means of assistance that could be widely available are curricular supports embedded within a curriculum, materials to support teacher discussion and reflection, videos of classrooms illustrating the higher levels, and Web-based teacher assistance programs that could support answers to teacher's questions and teacher interaction and support of each other. Systems of teacher professional development that could help teacher-learning communities themselves move through math-talk levels would develop the truly expert teachers needed in the 21st century.

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